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A NEW APPROACH TO TURBULENT BOUNDARY LAYER PROBLEMS

by Donald Ross

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A NEW APPROACH TO TURBULENT BOUNDARY LAYER PROBLEMS[1]

Donald Ross[2]

SYNOPSIS

A new approach is taken to the problem of incompressible, turbulent boundary layers. From physical considerations involving the diffusion of turbulent stresses, the boundary layer is divided into three regions distinguished by the relative proximity of the wall. Within a distance of about ten per cent of the boundary layer thickness from the wall, the local wall conditions have an important effect; whereas the outer seventy-five per cent is governed entirely by the spatial history of the boundary layer. The inner region includes the laminar sublayer, the buffer zone, and that part of the turbulent region characterized by the logarithmic velocity profile. The outer region of the turbulent boundary layer is similar to that of a wake or jet, being characterized by a sharp but irregular boundary between turbulent and non-turbulent regions. Here the velocity profile may usually be approximated by a three-halves power deficiency relation. The region between the inner and outer parts is a blending region more complicated than either of the other two and not amenable to simple analysis. This new physical analysis, combined with empirical correlations, leads to a new method for calculating two-dimensional turbulent boundary layers that usually only involves algebraic equations and is more versatile than any of the previously published methods.

INTRODUCTION

The basic differential equations for turbulent flows were derived by Osborne Reynolds(1)[3] in 1895 by assuming that all instantaneous velocities could be expressed as the sums of time-average velocities and instantaneous fluctuations about the average. Reynolds inserted his expressions for instantaneous velocities and pressures in the Navier-Stokes differential equations and took a time average. The resultant equations include additional terms, known as the Reynolds stresses, that involve the correlations of the fluctuation velocities and which represent the turbulent transport of momentum. When the Prandtl boundary-layer assumptions(2) are applied to these equations some of the remaining terms still involve the Reynolds stresses. It is because of these terms

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1. This paper is based on the thesis, "A Study of Incompressible Turbulent Boundary Layers," by Donald Ross, submitted to the Division of Applied Sciences of Harvard University in April, 1953, in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the field of Applied Physics. (Limited distribution as Ordnance Research Laboratory Technical Memorandum: ONR Project NR 062-139-1, June 1, 1953.)
2. Formerly Associate Professor of Engineering Research, Ordnance Research Laboratory, The Pennsylvania State University, State College, Pennsylvania; now Member of the Technical Staff, the Bell Telephone Laboratories, Whippany, New Jersey.
3. Numbers in parentheses refer to references listed at the end of the paper.

that mathematical solutions of turbulent-boundary-layer problems have not been possible, as they have been for the corresponding laminar flows.

In the past thirty years, various approaches have been taken to turbulent-boundary-layer problems, all of them without real success. The present status of the problem is that the theoreticians are awaiting the results of hot-wire measurements in turbulent shear flows with the hope that these experiments will reveal the basic mechanism of turbulent stresses and thus enable the building of a rigorous theory. It is believed by most workers in the field that it will be many years before such a theory is completely developed and still more years before it can yield results of practical value to the engineer.

In the meanwhile, as engineering problems require answers and cannot wait for the ultimate theoretical solution, it is desirable that improved semi-empirical methods of calculating turbulent boundary layers be developed. The present paper constitutes a report on a recent analytical study of turbulent boundary layers aimed at satisfying this need.[4] In this study, which was partially supported by the Office of Naval Research, the wealth of experimental data and theoretical ideas in the literature were reexamined and reevaluated in terms of some of the newer information and ideas concerning turbulent shear flows. One result was the development of a new method for predicting the behavior of two-dimensional boundary layers that is applicable whenever the initial spatial variation of the boundary layer is known.

Division of the Boundary Layer into Regions

The shear stress in a turbulent flow is composed of viscous and turbulent terms, and it may be written:

$$\frac{\tau}{\rho} = (\nu + \epsilon)\frac{\partial u}{\partial y} \qquad (1)$$

where ν is the molecular viscosity and ϵ the eddy viscosity associated with the turbulent fluctuations. The usual approach to the turbulent boundary layer is in terms of the relative influence of the molecular and eddy viscosities. Thus, very close to the wall there is the laminar sublayer in which the molecular viscosity predominates and in which the velocity is proportional to the distance from the wall. Further out the molecular viscosity has a negligible effect and the eddy viscosity governs the flow. In between, there is a buffer zone in which both viscosities must be considered. Except at the lowest Reynolds numbers, the laminar sublayer and the buffer zone are very small. One is therefore left with a single turbulent region and solutions of the equations of motion are sought in which the molecular viscosity is practically neglected.

Usually a single function is used to describe the entire turbulent velocity distribution. The well-known power law is an example of such a formula. Sometimes it gives a reasonable fit, but usually it is a poor approximation. Superposition formulas developed by Hudimoto(3) in 1935, and more recently by Rotta(4) and Ross and Robertson(5), also fail to agree with the details of the velocity profiles. Specifically, none of these single functions admits the existence of inflections in the profiles; and yet, such inflections are invariably found for profiles close to separation.

Recognizing that the usual analyses do not achieve sufficient agreement with the details of measured velocity profiles, the physical factor was sought that

4. Ross, Ph. D. Thesis, Ibid.

would account for the experimentally observed features of turbulent boundary layers. This factor is the relative proximity to the wall. In the new analysis, there is superimposed on the usual division according to viscosity considerations a division into three regions according to the relative influence of local wall conditions. These zones are approximately the inner ten per cent of the boundary layer thickness, in which the proximity of the wall influences the flow; the outer seventy-five per cent, governed by the spatial history of the boundary layer; and, a blending zone between the others in which the flow is influenced by both the wall proximity and the spatial history. Fig. 1 shows a typical velocity profile and illustrates the superposition of the two methods of dividing the boundary layer. The inner region is seen to include the laminar sublayer and the buffer zone as well as an inner turbulent part.

To understand the reasoning behind this method of division it is essential to realize the fundamental distinction between laminar and turbulent shear flows. In laminar flows the property that gives rise to the shear stress, namely the viscosity, exists throughout the fluid in a uniform manner. The laminar boundary layer exists because the physical boundary condition of zero velocity at a solid surface differs from the potential theory solution for the main fluid flow. In the case of the turbulent boundary layer, the high intensity turbulence that supports the shear has its origin very close to the wall and is diffused into the fluid. Because the turbulence originates very close to the wall, in the region of very high velocity gradients, and diffuses out into the moving fluid, it follows that the turbulence intensities and shears in the outer part of the boundary layer are governed by the spatial history of the flow while those near the surface are more a function of local conditions. This concept of the space-history of the flow was expressed by Schultz-Grunow(6) in 1938 and again by Dryden(7) in 1946. Prandtl(8) was aware of this important point as early as 1935, when he stated, "for each velocity profile develops in some regular manner from the velocity profiles situated further upstream; i.e.....depends on the previous history of the portion of the fluid considered." This concept of the diffusion of turbulent stresses and of the importance of spatial history is behind the division of the boundary layer into the zones shown in Fig. 1.

The properties of the inner region are closely related to the presence of the wall. Here occurs the majority of the conversion of mean motion to turbulence and of turbulence to heat. The production of turbulence predominates over its dissipation. In the outer region, both production and dissipation occur at very much reduced rates, with dissipation predominating. The outer region is also characterized by an "intermittency effect" that arises from the irregular nature of the outer boundary of the turbulent region(9)

Between the inner and outer regions there is a blending region that is partially dependent on local conditions and partially on spatial history. This blending region combines the features of the other two regions and is necessarily more complex than either of the others. It is not to be confused with the "overlap" region. In analyses of equilibrium boundary layers by Millikan(10) and von Mises(11) the velocity profile is considered from the wall outward and from the outside in. It is also assumed that there is an overlap region close to the wall in which both functions are equally valid. As the overlap region is closer to the wall than the blending region, the overlap condition can only be used when the outer profile is known through the blending region. Thus, the overlap concept has some apparent value when one is dealing with equilibrium boundary layers, but it is of no value when dealing with the more general non-equilibrium case.

Velocity Functions

If the methods of dimensional analysis are applied to the problem of turbulent-boundary-layer velocity profiles, using the concepts already developed, two functions result.(12) The one for the inner profile:

$$\frac{u}{u_*} = f\left(\frac{yu_*}{\nu}, \frac{y}{k}, \frac{\nu}{\rho u_*^3}\frac{\partial p}{\partial x}\right) \tag{2}$$

involves the wall-shear-stress in the definition of the friction velocity

$$u_* = \sqrt{\frac{\tau_w}{\rho}}, \tag{3}$$

the wall roughness k in the term y/k, and the local pressure gradient in the last term. The relation for the outer profile:

$$\frac{u_1 - u}{u_1} = g\left(\frac{\delta - y}{\delta}, \text{spatial history}\right) \tag{4}$$

expresses the velocity deficiency in terms of the relative distance from the outer edge and the spatial history. (u_1 is the free-stream velocity and δ is the effective boundary layer thickness.)

The inner function consists of three parts: the laminar sublayer, the buffer zone, and the inner turbulent region. The first two of these are usually quite thin. The inner turbulent region is characterized by a logarithmic velocity profile, which can be derived directly from dimensional analysis by assuming that a most important influence on the eddy viscosity is the distance from the wall. The equation, of the form

$$\frac{u}{u_*} = A + B \log \frac{yu_*}{\nu}, \tag{5}$$

is well verified by measured velocity profiles. The Law of the Wall proposed by Ludwieg(13) in 1949 states that for smooth walls the coefficients A and B of the logarithmic inner profiles are practically universal constants independent of pressure gradients, of the Reynolds number, and of the past history of the flow. (The functional relation of equation 2 includes a pressure gradient term, but this may be shown to be small except for low Reynolds numbers and or extreme pressure gradients.) The author has examined[5] approximately one dozen experiments in which both wall shear stress and velocity profiles were measured and has found no appreciable difference of the coefficients for the various types of flows. For all practical purposes, the coefficients A and B are both 5.6, and the formula for the inner turbulent region is:

$$\frac{u}{u_*} = 5.6 + 5.6 \log_{10} \frac{yu_*}{\nu} \tag{6}$$

This formula is valid for $\frac{yu_*}{\nu} > 20$ and $y/\delta < 0.1$.

For the outer turbulent region, a simple formula which gives an excellent fit to most velocity profiles is the three-halves power deficiency law:

5. Ross, Opus Cit.

$$1 - \frac{u}{u_1} = D\left(1 - \frac{y}{\delta}\right)^{3/2} \tag{7}$$

This formula, which is valid for approximately the outer seventy-five per cent of the boundary layer, is similar to that proposed by Darcy, nearly a century ago, for the central region of pipe flow; hence the choice of the letter D for the coefficient. Its local value, which depends on the spatial history of the flow, varies from about 0.3 for flatplates to as high as 1.3 near separation.

The various assumptions that have been used to derive the inner and outer velocity functions fail in the blending region between these two regions. The physics of this region is not such that the velocity profile can be described by any simple functional relation; neither wall conditions nor spatial history can be ignored. An infinite number of outer profiles may exist. For each outer profile a variety of inner profiles may exist, depending primarily on the local Reynolds number. The velocity profile in the blending region will depend upon the combination of inner and outer profiles. For those cases for which the extrapolated inner and outer functions cross each other at an appreciable angle, the curve in the blending region will include an inflection point. Some of the possible combinations are illustrated in Fig. 2, which shows clearly the inflections that occur for high D profiles.

The division of the boundary layer into inner and outer regions, superimposed on the normal division into viscous and turbulent zones, has resulted in a somewhat more complex picture of the turbulent boundary layer than that usually considered. That this additional complexity is worthwhile is supported by the simplifications that result in the analysis of each of the regions. Fig. 3 illustrates the division of the boundary layer into regions and summarizes their properties. Along with the extents of the regions and the types of velocity profiles, the general characteristics of the shear stress profile and of the Prandtl mixing length (14) are indicated. These are based on experimental observations and are included because of the application of Prandtl's momentum-transfer theory at a later stage in the analysis. Fig. 4 shows a typical set of experimental profiles, measured by Schubauer and Klebanoff.(15) The solid curves are the inner and outer profiles given by equations 6 and 7. The curves progress from equilibrium, flat-plate conditions practically to separation. The blending between the two portions is clearly illustrated.

The strongest argument for the present analysis is that it is the simplest way of treating the turbulent boundary layer consistent with known physical properties that leads to useful results. The remainder of the present paper is devoted to a brief discussion of the new method for calculating two-dimensional turbulent boundary layers that was developed from this analysis.

Calculation Method

When the turbulent boundary layer problem is expressed in its most general form there are four simultaneous equations, two of which may be non-linear differential equations and all of which are interdependent. These equations are:

1) the growth of the momentum thickness as a function of the wall shear stress, pressure gradient, and a shape parameter;
2) the shape parameter of the velocity profile as a function of the pressure distribution and the spatial history of the flow;
3) the wall shear stress as a function of the shape of the velocity profile and the momentum-thickness Reynolds number; and,
4) the detailed velocity distribution as a function of the shape parameter,

the boundary-layer thickness, and the local wall shear stress. Obviously, simultaneous solution of these four equations is impractical. What is required is an approximate method whereby the four equations can be solved separately and, if necessary, second approximations made. The analysis presented here makes such a solution possible for a wide variety of two-dimensional boundary layers.

The boundary layer at any station is calculated as follows:

1) The momentum thickness if found as a function of the pressure distribution, through a recently developed approximate integration of the von Kármán momentum equation.
2) The coefficient D of the three-halves power outer velocity distribution replaces the shape parameter. A function of this coefficient is related to the spatial history by a semi-empirical equation derived from Prandtl's momentum-transfer theory. This function then leads to D itself through an empirical correlation.
3) The wall-shear-stress coefficient is related to D and the local momentum-thickness Reynolds number by Ludwieg's "Law of the Wall," and by a second empirical correlation.
4) Finally, the velocity distribution at the station is calculated in two parts. The outer part is a three-halves power deficiency curve having coefficient D. The inner part is logarithmic with its magnitude determined from the wall-shear-stress coefficient. A smooth curve is used to blend the two regions.

The method applies to redistributing flows that proceed from separation towards flat-plate equilibrium; as well as to the usual boundary layer in an adverse pressure gradient that goes from equilibrium towards separation. The occurrence of separation is also correlated with the coefficient D.

Growth of the Momentum Thickness

The equation that is almost invariably used for the growth of the momentum thickness is the von Kármán integral momentum equation:

$$\frac{d\theta}{dx} = \frac{c_f}{2} + (2 + H)\frac{\theta}{\rho u_1^2}\frac{dp}{dx} \tag{8}$$

where c_f is the wall-shear-stress coefficient and H is the common shape parameter. This equation, in somewhat different form, was originally derived by von Kármán(16) in 1921 for use with laminar boundary layers. Although it is extensively applied to turbulent-boundary-layer problems in this form, actually for many turbulent flows the contributions of the turbulent normal stresses must be included as an additional term. The exact solution of the corrected equation requires the simultaneous solution of the equations for the wall-shear-stress coefficient and for the shape parameter. However, Ross and Robertson(17) have recently developed an approximate solution that may be used directly. They distinguish between two cases: 1) small pressure gradients, for which the boundary layer is close to the equilibrium condition, and 2) large adverse pressure gradients. In the first case, step-by-step solution of equation 8 is utilized, with both c_f and H assumed constant for each step and each usually having the equilibrium, flat-plate value. For comparatively large pressure gradients, they develop a simple algebraic, power expression:

$$\frac{\theta}{\theta_i} = \left(\frac{u_{1_i}}{u_1}\right)^{2+G} \tag{9}$$

which gives good agreement with experimental results. The exponent, 2 + G, is a function of the momentum-thickness Reynolds number at the initial station, varying from about 5.0 at transition ($R_\theta \sim 10^3$) to about 4.2 at a high momentum-thickness Reynolds number of about 10^5 as shown in Fig. 5. Utilizing this result, the momentum thickness for any station may be readily computed without first solving the other three equations, and without knowledge of the shape parameter. Usually a direct algebraic solution will be possible for θ as a function of only the potential velocity, u_1, or pressure, distribution. The momentum-thickness Reynolds number is then given by:

$$R_\theta = \frac{\theta u_1}{\nu} \cong \left(\frac{u_{1_i}}{u_1}\right)^{1+G} R_{\theta_i} \tag{10}$$

The wall shear stress will be shown to be partially a function of this Reynolds number.

Behavior of the Shape Parameter

In most of the published analysis methods a single-parameter family of velocity profiles is assumed and a differential equation is written for the variation of the shape parameter with distance. This auxiliary equation must be solved simultaneously with the momentum equation and the equation for the wall shear stress. In the present development, the boundary layer is divided into regions and no single parameter characterizes the entire velocity profile. However, the inner profile, as well as the outer one, is dependent on the outer shape parameter D, so that solution of an equation for D is the analogous step in the calculation procedure.

The fundamental physical property governing D is the dependence of the outer velocity profile on the spatial history of the flow and its independence of local wall conditions. The problem of expressing this by an equation was attacked through use of the Prandtl momentum-transfer theory. Although this mixing-length theory has long been held in contempt by many theoreticians, recent investigations[18, 19] based on the statistical aspects of turbulence have shown that it is a good approximation for the calculation of velocity profiles.

Prandtl's theory gives a differential equation for the velocity profile as a function of the shear-stress and mixing-length profiles:

$$\frac{\tau}{\rho} = \ell^2 \frac{\partial u}{\partial y} \left| \frac{\partial u}{\partial y} \right| \tag{11}$$

A typical shear stress profile is shown in Fig. 6. The outer region is essentially a linear function[20] with intercepts at an effective (outer) wall shear stress, τ_{w_e}, and at the effective boundary-layer thickness, ℓ. The mixing length, δ, is assumed to be constant in the region outside of the influence of the wall. When the linear shear stress profile and constant mixing length are assumed, Prandtl's theory yields the 3/2-power velocity deficiency relation with

$$D = \frac{2}{3}\frac{\delta}{L}\sqrt{\frac{\tau_{w_e}}{\rho u_1^2}} \tag{12}$$

where L is the constant mixing length and τ_{w_e} and δ are defined in Fig. 6. By simple algebraic manipulation one can create a parameter that should be a function of distance but not of local conditions. This combination is:

$$\frac{\delta}{D}\left(\frac{u_{1_i}}{u_1}\right) = \frac{3}{2} L \left(\frac{\tau_{we}}{\rho u_{1_i}^2}\right)^{-1/2} \approx f(x) \tag{13}$$

where u_{1_i} is measured at the initial station. It can be reasoned that if the outer region of a turbulent boundary layer is similar to the shear flow in a wake or jet, then L and τ_{we} will vary with distance in a manner independent of changes in local wall conditions and of pressure gradients.

Experimental confirmation of equation 13 is found in Fig. 7, which is a plot of the left-hand side of equation 13 for the data published by Schubauer and Klebanoff.[15] In this experiment the flow had practically zero pressure gradient up to the 18-ft station and a strong adverse pressure gradient after this point. (Separation occurred just ahead of the 26-ft station.) The linear trend that was established in the constant pressure region continued into the strong adverse pressure gradient. Another confirmation, this time for a redistributing flow having constant outer velocity, is shown in Fig. 8. Here the data of Klebanoff and Diehl[21] for the flow along a smooth plate preceded by two feet of very rough surface shows the same linear trend. This confirms a basic assumption of the present development, that all turbulent flows obey the same general relations. Most of the published analyses are particularly unsuccessful in calculating the shape parameter for this type of flow. In the original study[6] several other examples of normal and of redistributing flows are presented, and it is concluded that, for a distance equal to about fifty initial-boundary-layer thicknesses, the outer velocity profile is governed by a relation of the form:

$$\frac{\delta}{D}\left(\frac{u_{1_i}}{u_1}\right) = A\,(x - x_i) + \frac{\delta_i}{D_i} \tag{14}$$

provided the flow is very closely two-dimensional. Following the linear region, the initial character of the outer flow is pretty much submerged in the turbulence generated within this distance and a new equilibrium is established.

The coefficient A may be calculated for those cases for which an equilibrium boundary layer is established in a region of zero pressure gradient ahead of a region of relatively strong pressure gradient, or of change in wall conditions. As A has a constant value, it may be calculated from the equilibrium, constant-pressure, flow:

$$A = \frac{d}{dx}\left(\frac{\delta}{D}\,\frac{u_{1_i}}{u_1}\right)_e = \left(\frac{\delta/\theta}{D}\right)_e \frac{d\theta}{dx} \cong \left(\frac{\delta/\theta}{D}\right)_e \left(\frac{c_f}{2}\right)_e \tag{15}$$

For equilibrium profiles, all of the quantities are related, and it is possible[6] to relate A simply to the value of the wall-shear-stress coefficient:

$$A \cong 0.025\,(1 + 12\sqrt{c_f}\,) \tag{16}$$

which is in turn a function of the momentum-thickness Reynolds number, and/or the relative roughness.

In cases where A is known, the value of D at any station can be calculated if δ, the thickness of the boundary layer, is known. The momentum equation yields θ, so that the ratio of δ to θ is sufficient. If the blending region were fully understood, it might be possible to calculate this ratio; however, it was

6. Ross, Opus Cit.

necessary to resort to empirical correlation.

In the original study[7], over one-hundred experiments were examined, and eighteen were selected for the empirical analysis. These experiments were selected as the best representatives of a wide variety of experimental conditions. It being a logical contention of the present analysis that all types of turbulent-boundary-layer flows must be governed by the same physical laws, at least one experiment for each type of two-dimensional configuration was chosen. The correlation of δ/θ with D is shown in Fig. 9. From this result, δ can be calculated if D is known. Actually, returning to equation 14, it is D that is to be found. Equation 14 may be written in the form:

$$\frac{\delta/\theta}{D} = \left[A\left(\frac{x-x_i}{\theta_i}\right) + \left(\frac{\delta/\theta}{D}\right)_i\right]\left(\frac{u_1}{u_{1_i}}\right)\left(\frac{\theta_i}{\theta}\right) \tag{17}$$

where the left-hand side is the function of D given in Fig. 10. Thus, from equation 17 and Fig. 10, the value of D at any station may be found provided that the value of A and the initial conditions are known. The above method of computing D lays stress on the initial spatial variation of the outer boundary layer. It follows that boundary layers with different initial spatial histories (different A's) will behave quite differently when acted on by the same pressure distribution.

The Wall Shear Stress

In equation 6, the inner velocity profile is given as a function of the friction velocity. A useful expression for this latter quantity results from writing the inner velocity equation for the momentum thickness distance. The equation for this fictitious effective velocity, u_{θ_e}, is simply:

$$\frac{u_{\theta_e}}{u_*} = 5.6 + 5.6 \log \frac{\theta u_*}{\nu} \tag{18}$$

It can be manipulated into the expression:

$$\sqrt{\frac{2}{c_f}} = \frac{u_1}{u_*} = \frac{u_1}{u_{\theta_e}}\left[5.6 + 5.6 \log\left(R_\theta \sqrt{\frac{c_f}{2}}\right)\right] \tag{19}$$

This equation may be solved numerically for u_* as a function of R_θ and u_{θ_e}/u_1. The result is found to fit very closely the approximate expression:

$$\frac{u_1}{u_*} \equiv \sqrt{\frac{2}{c_f}} \cong \frac{u_1}{u_{\theta_e}}\left[0.7 + 5.0 \log\left(R_\theta + \frac{u_{\theta_e}}{u_1}\right)\right] \tag{20}$$

It should be noted that this formulation of the wall friction relation for turbulent boundary layers follows directly from the universal Law of the Wall proposed by Ludwieg.[13] The friction analysis given here parallels that of Ludwieg with the difference being that the momentum-thickness velocity will be related to the new outer parameter D instead of to the usual shape parameter H. The empirical correlation of u_{θ_e}/u_1 with D is shown in Fig. 11. The scatter of the data is somewhat greater than that for δ/θ, but no consistent trend could be found with any other variable and the scatter is attributed less

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7. Ross, Opus Cit.

to second order effects than to experimental error. Fig. 12 enables the rapid estimation of the wall-shear-stress coefficient as a function of the coefficient D and the momentum-thickness Reynolds number, R_θ.

The Velocity Distribution

The velocity distribution is made up of the separate inner and outer profiles connected by a blending curve. Equation 6 for the inner profile may be written:

$$\frac{u}{U_1} = \frac{u_{\theta c}}{U_1}\left[1 + \frac{\log\left(\frac{y}{\theta}\right)}{1 + \log\left(R_\theta\sqrt{\frac{c_f}{2}}\right)}\right] \tag{21}$$

Where all of the required quantities have already been computed. Similarly, the outer profile is given directly by equation 7 in terms of known quantities. A smooth curve may be used to connect the curves in the blending region. Thus the present analysis leads readily to the computation of the velocity distribution at a station, without requiring the simultaneous solution of a pair of differential equations.

Separation

The method of the present paper is particularly well suited to the prediction of separation. The calculation procedure gives a way of determining D, and the occurrence of separation may be correlated with D. Fig. 11 shows a linear variation for the velocity at the momentum thickness as a function of D. This extrapolates to zero at D equal to about 1.45, establishing an upper limit. The actual value will depend somewhat on the Reynolds number and on the downstream flow conditions. A good criterion for separation is $D = 1.3 \pm 0.1$.

CONCLUSION

In the present paper, a new approach to turbulent-boundary-layer problems has been outlined. The analysis considers that a most important factor governing the velocity profile is the relative proximity of the wall. Consideration of this factor has led to the division of the boundary layer into three regions. The properties of the inner, wall, region depend on local wall conditions and on the outer region. The latter is determined as a function of the spatial history of the flow. The resultant semi-empirical calculation method is the only one developed to date that applies to redistributing flows as well as to the usual case of flow in an adverse pressure gradient. The calculation procedure is also mathematically more simple, as the usual simultaneous differential equations are replaced by algebraic equations. This means that analytic answers are now possible for many problems previously requiring numerical solution. The method also yields a simple criterion for predicting separation. Its most serious limitation is that, in the algebraic form, it only applies for a distance equal to about fifty initial boundary layer thicknesses.

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17. "An Empirical Method for Calculation of the Growth of a Turbulent Boundary Layer," Donald Ross and J. M. Robertson, Journal of the Aeronautical Sciences, Readers' Forum, Vol. 21, p. 355, May, 1954.

18. "Studies of von Kármán's Similarity Theory and its Extension to Compressible Flows. A Critical Examination of Similarity Theory for Incompressible Flows," C. C. Lin and S. F. Shen, NACA Technical Note 2541, November, 1951.

19. "The Eddy Viscosity in Turbulent Shear Flow," A. A. Townsend, Philosophical Magazine, Vol. 41, 1950, pp. 890-906.

20. "Shear Stress in a Turbulent Boundary Layer," Donald Ross and J. M. Robertson, Journal of Applied Physics, Vol. 21, 1950, pp. 557-561.

21. "Some Features of Artificially Thickened Fully-Developed Turbulent Boundary Layers with Zero Pressure Gradient," by P. S. Klebanoff and Z. W. Diehl, NACA Technical Note 2475, October 1951.

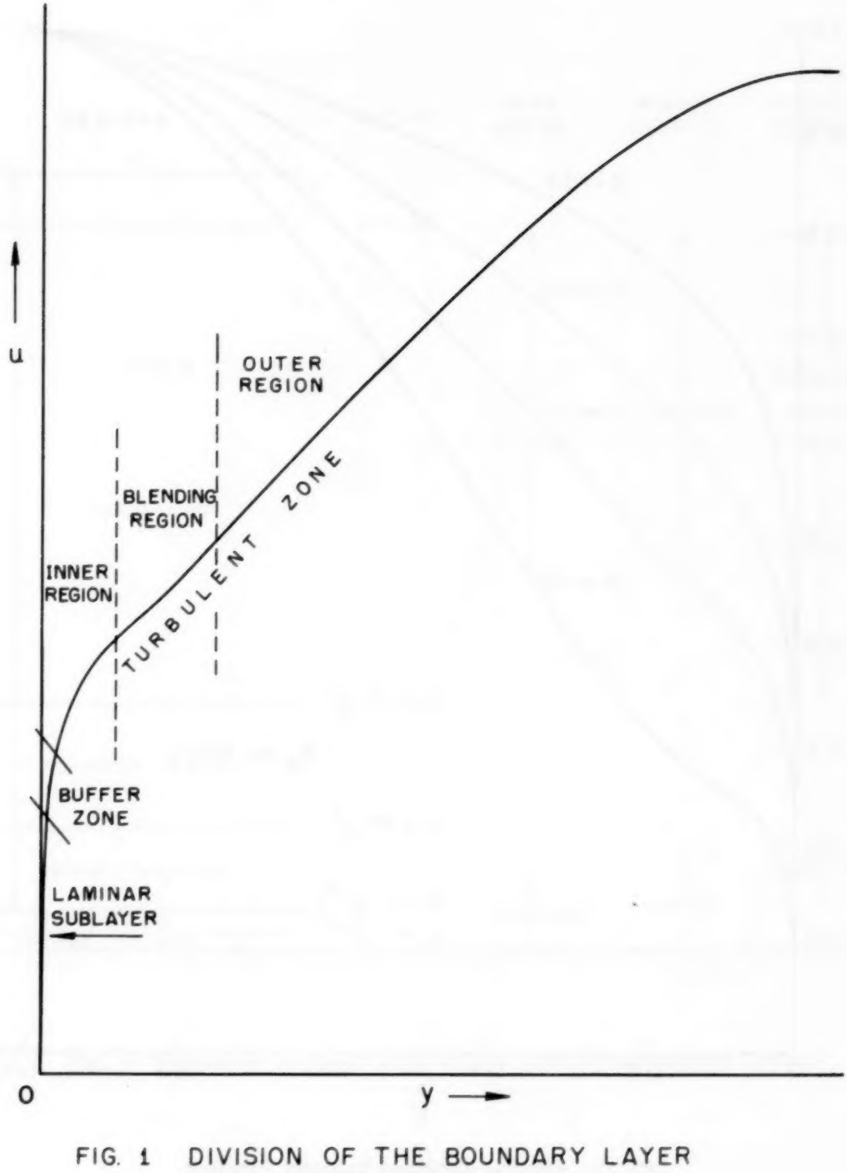


FIG. 1 DIVISION OF THE BOUNDARY LAYER INTO REGIONS

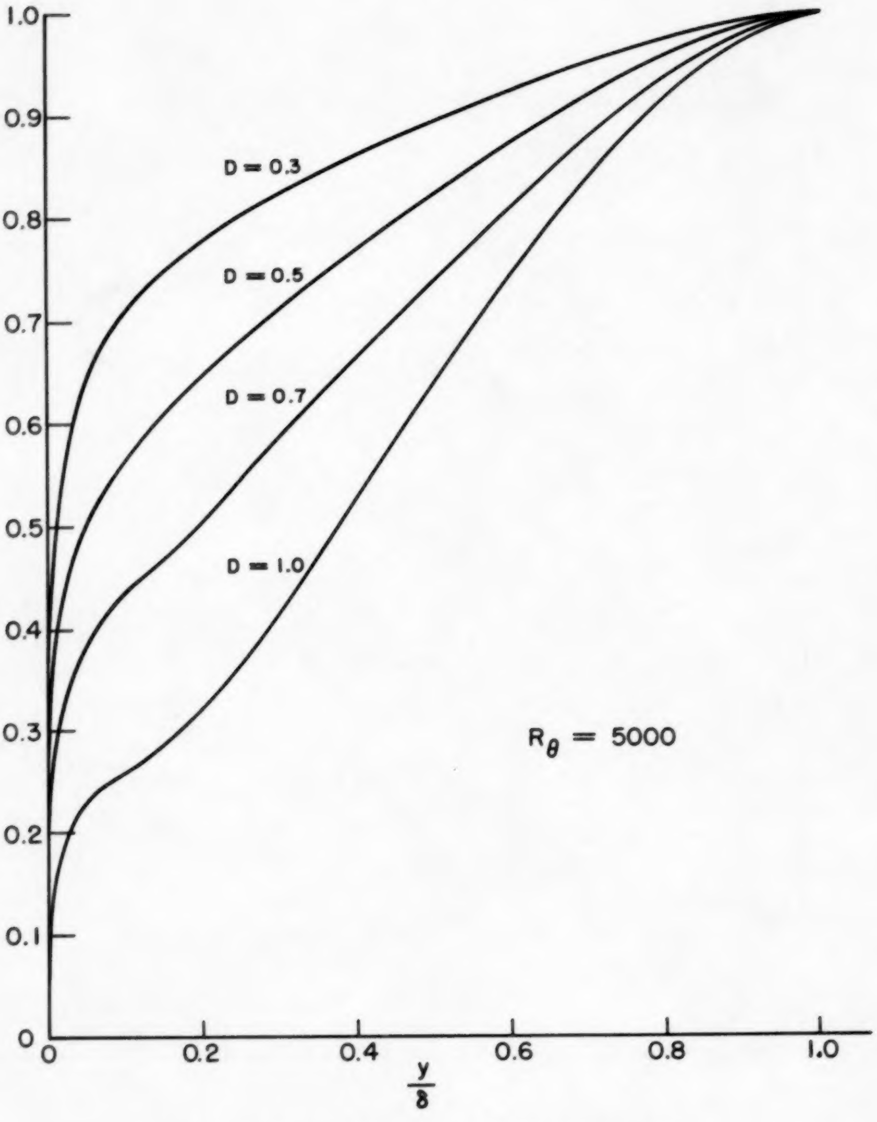


Fig. 2. Typical Idealized Velocity Profiles.

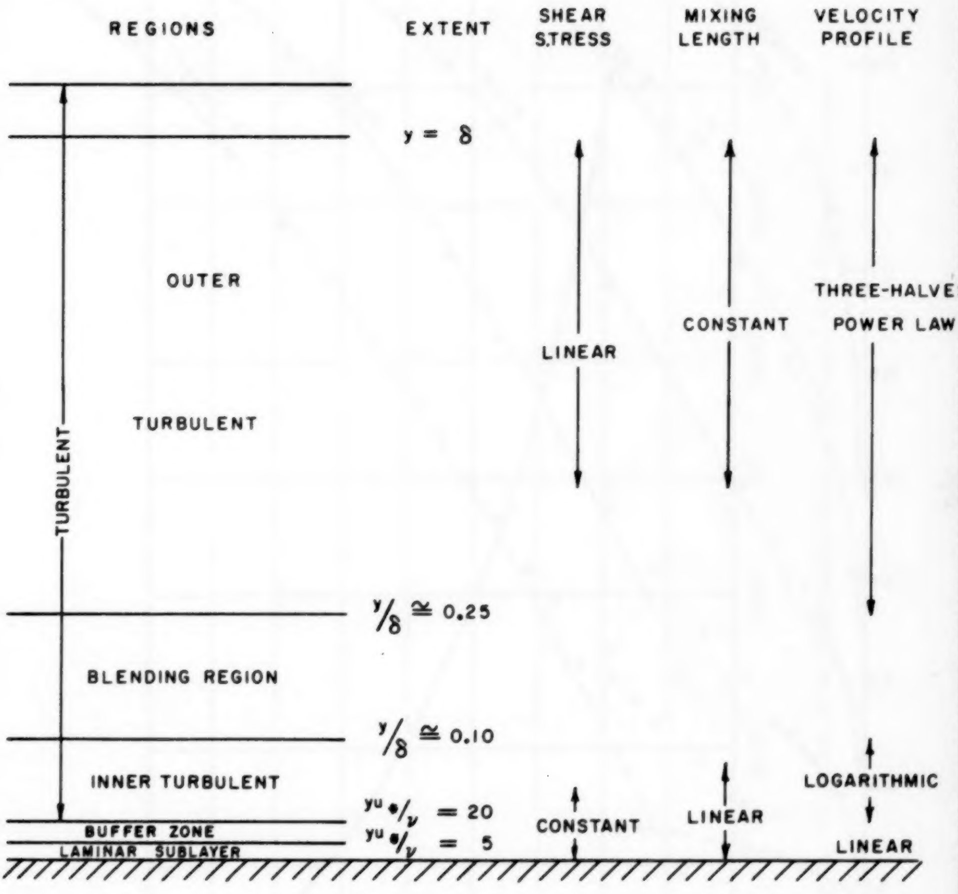
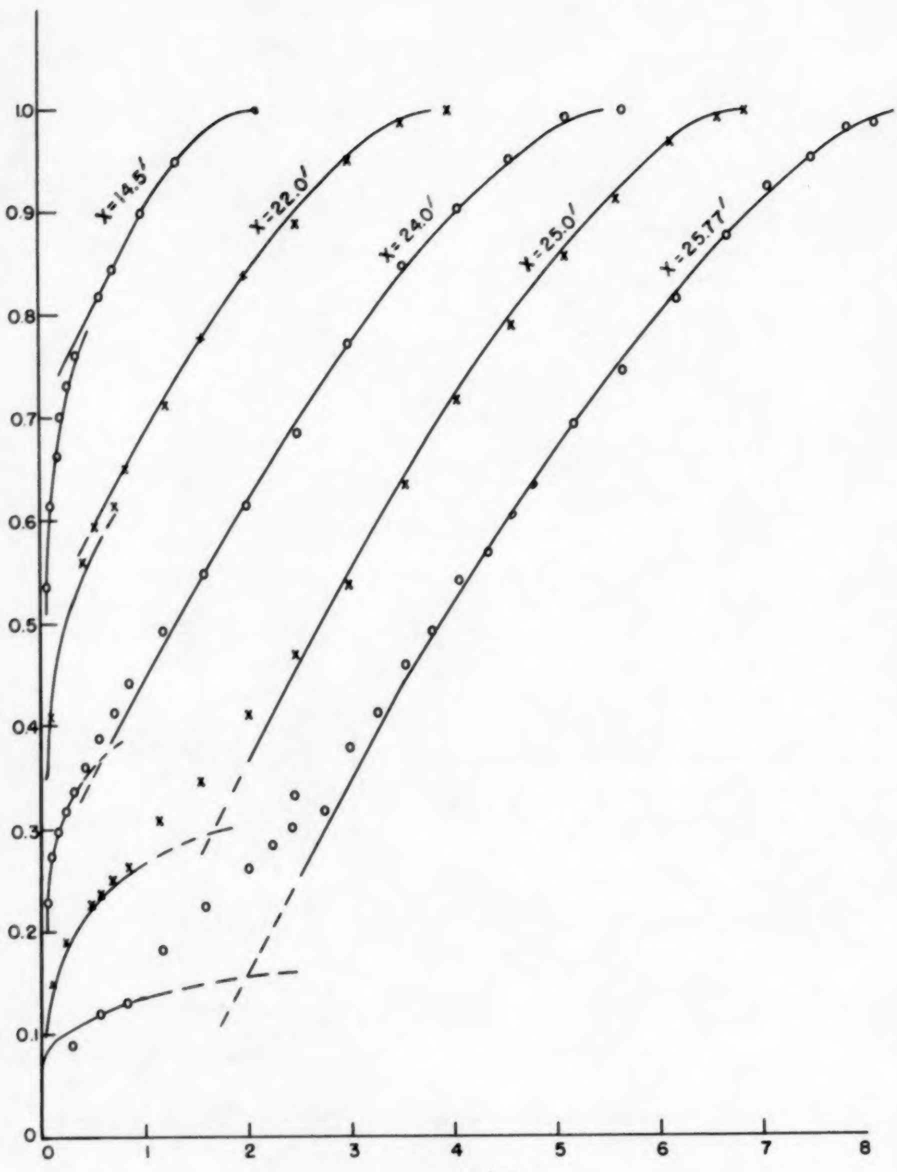


Fig. 3. Properties of the Regions of the Turbulent Boundary Layer.



TYPICAL BOUNDARY - LAYER TRAVERSES

Fig. 4. Measured by Schubauer and Klebanoff[15]

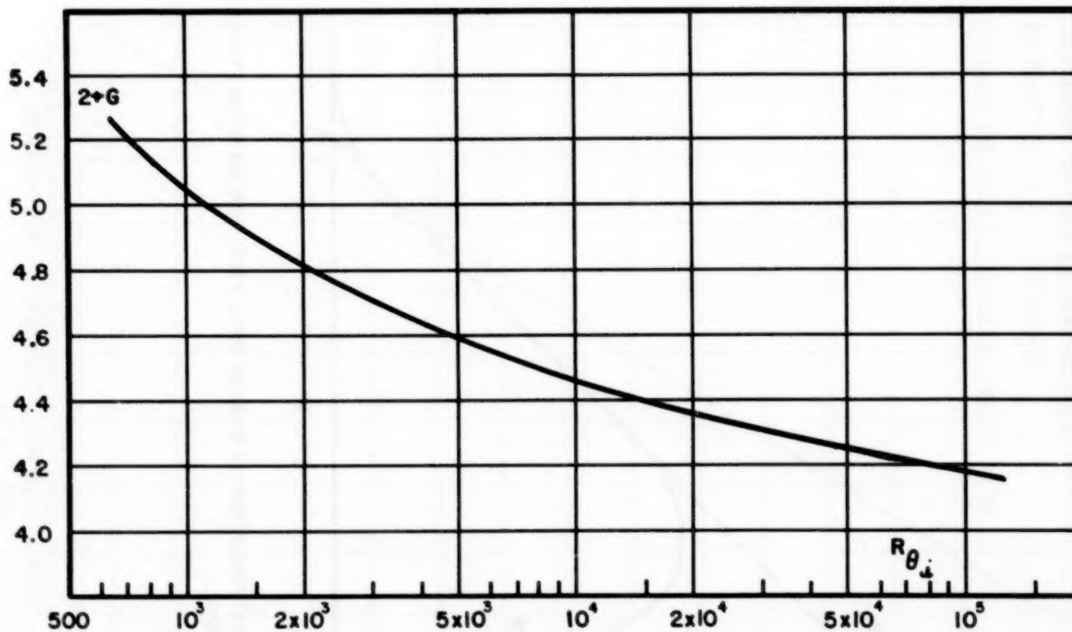


Fig. 5. Variation of the Momentum Equation Exponent with Reynolds Number.

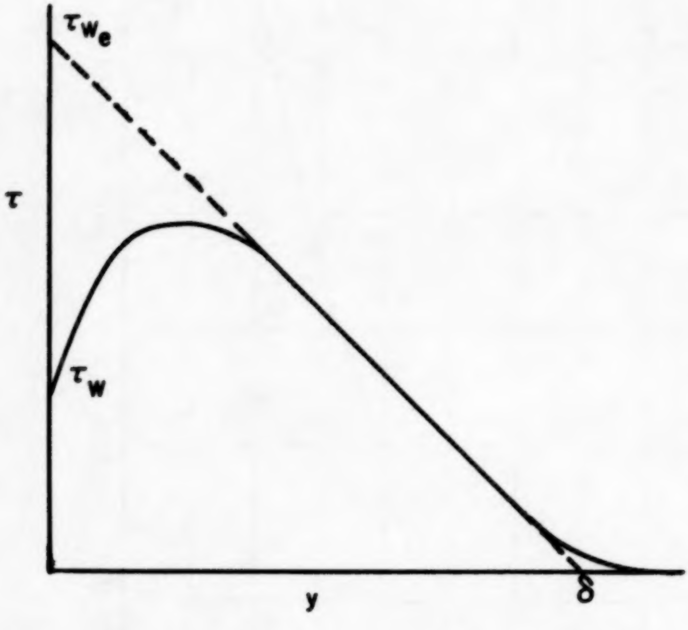


Fig. 6. Typical Shear-Stress Profile for a Flow with Adverse Pressure Gradient.

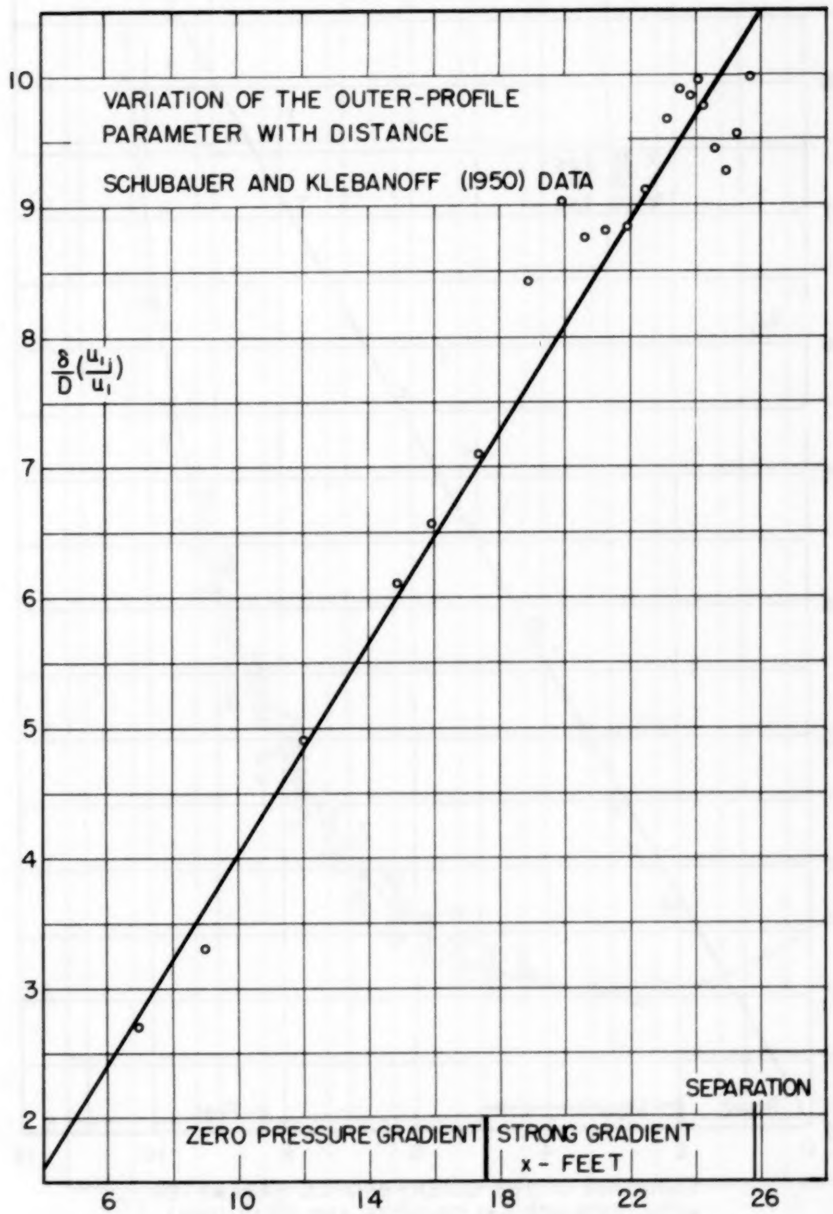
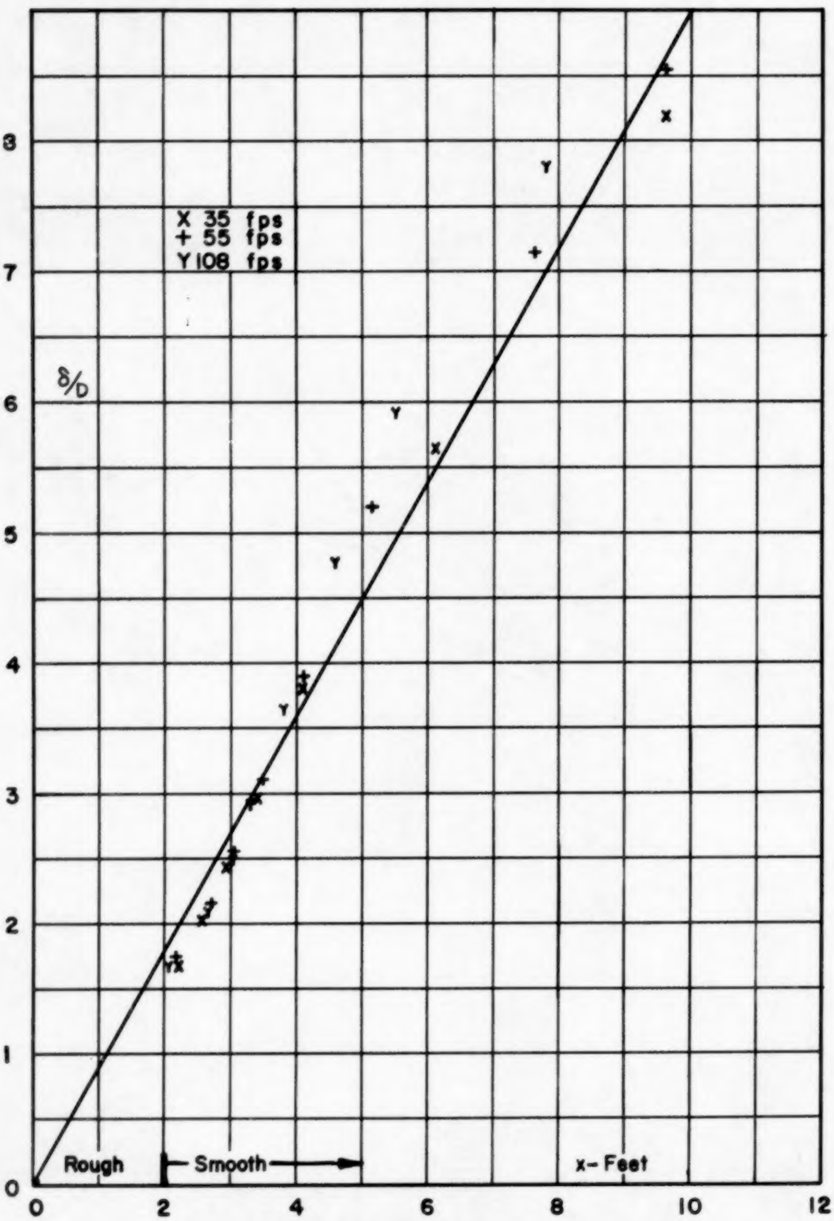

VARIATION OF THE OUTER-PROFILE
PARAMETER WITH DISTANCE
SCHUBAUER AND KLEBANOFF (1950) DATA
$\frac{\delta}{D}\left(\frac{u_{1j}}{u_1}\right)$
10
9
8
7
6
5
4
3
2
6
10
14
18
22
26
ZERO PRESSURE GRADIENT
STRONG GRADIENT
x - FEET
SEPARATION

Fig. 7.



VARIATION OF THE OUTER-PROFILE PARAMETER WITH DISTANCE—KLEBANOFF AND DIEHL (1951) SANDPAPER PLATE.

Fig. 8.

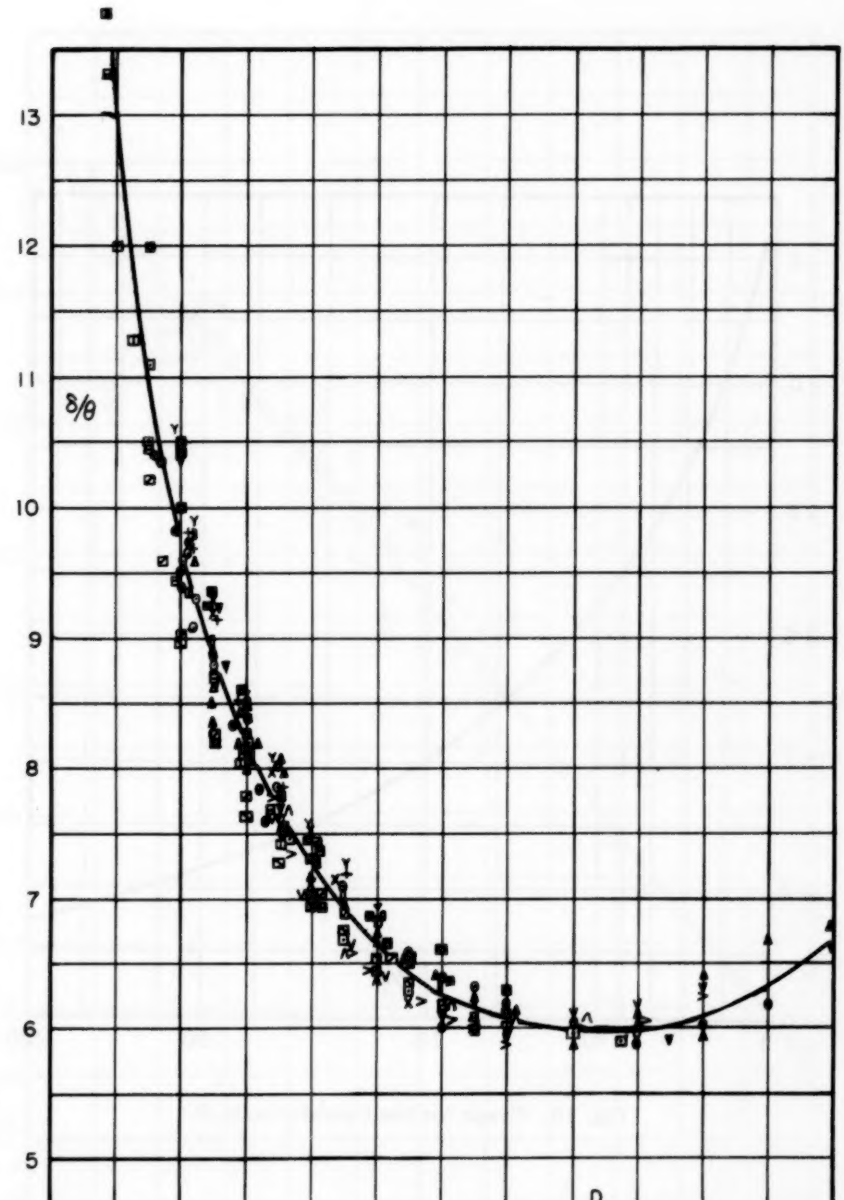


Fig. 9. Correlation of δ/θ as a Function of D.

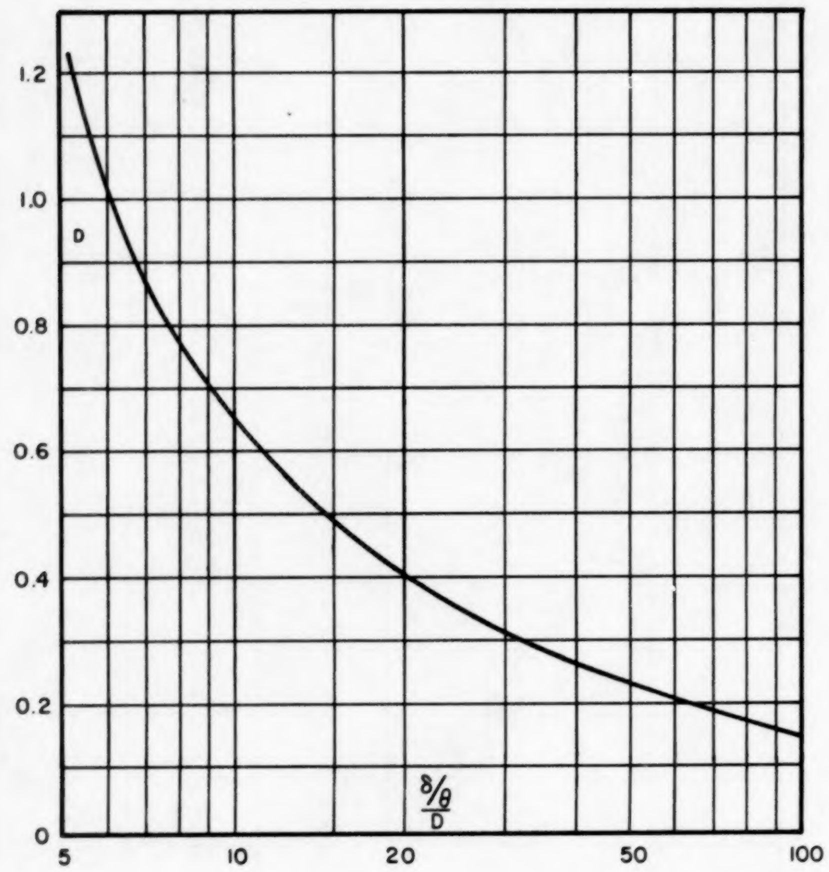


Fig. 10. Graph for the Calculation of D.

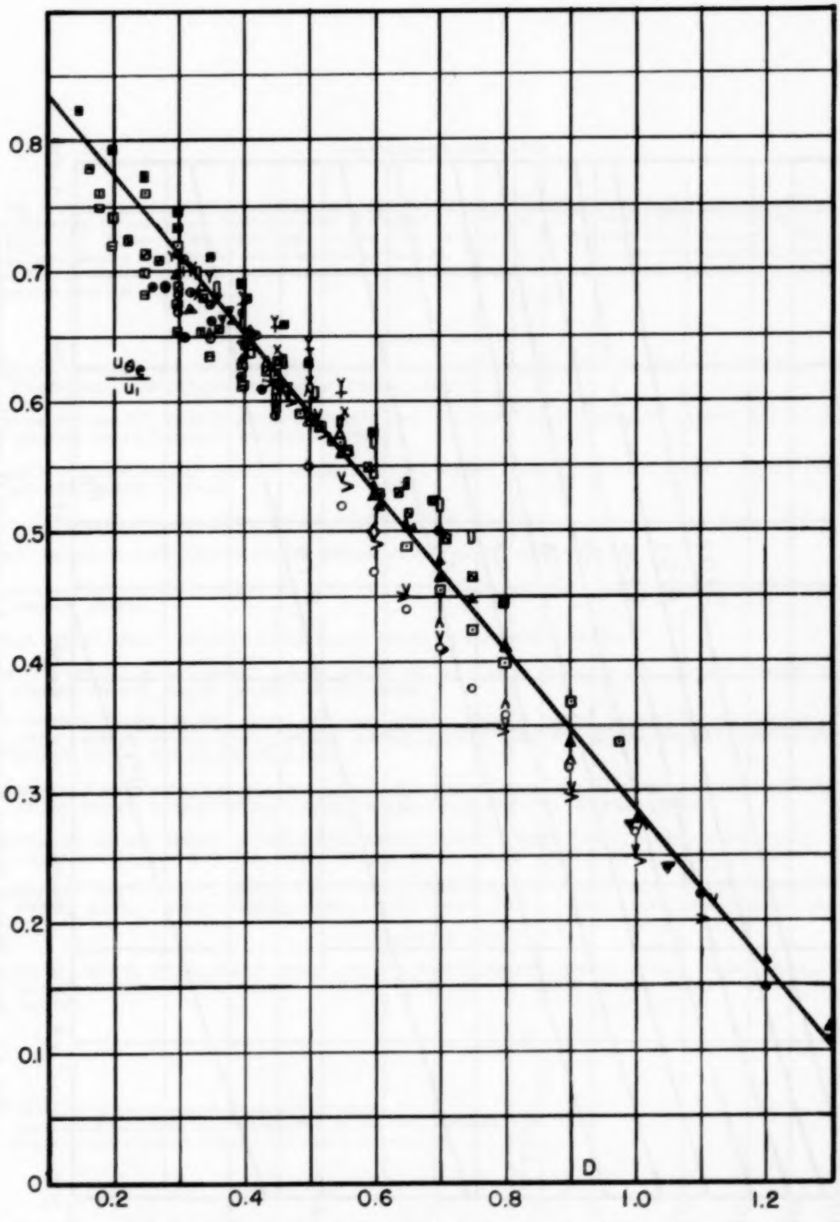


Fig. 11. Correlation of Effective Relative Velocity at Momentum-Thickness as a Function of D.

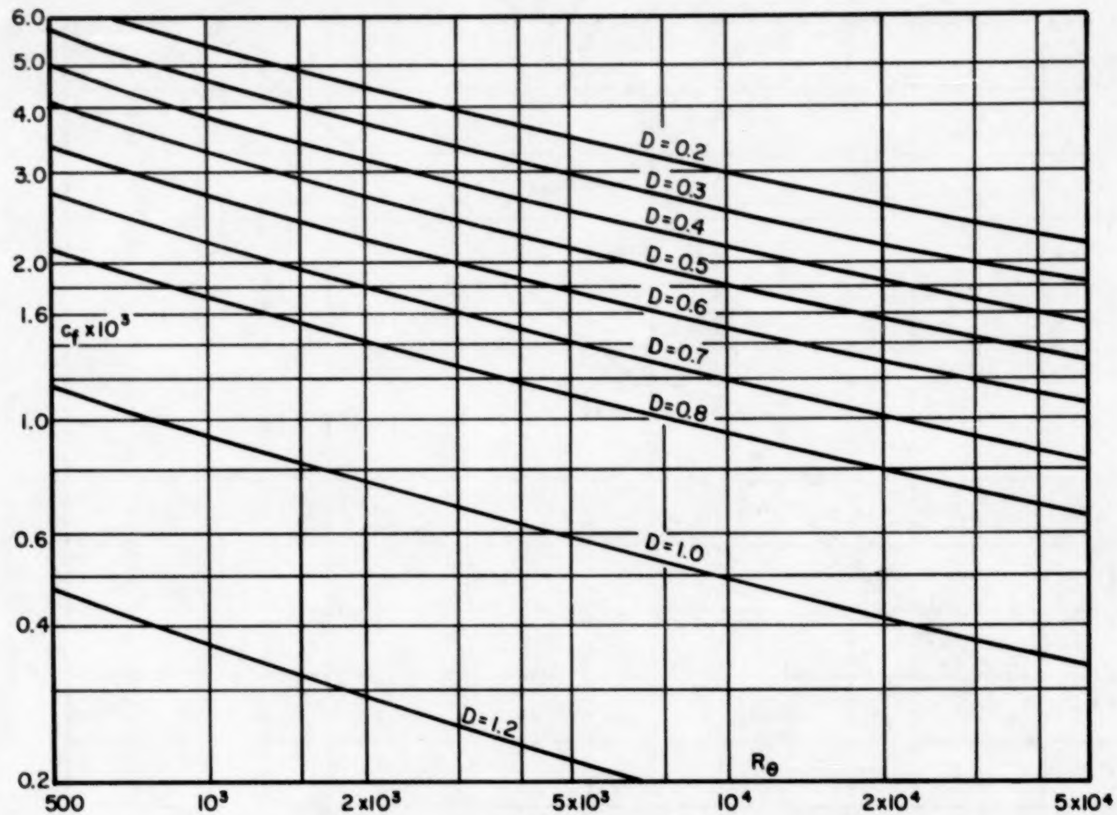


Fig. 12. Wall-Shear-Stress Coefficient as a Function of D and the Momentum-Thickness Reynolds Number.

PROCEEDINGS-SEPARATES

The technical papers published in the past year are presented below. Technical-division sponsorship is indicated by an abbreviation at the end of each Separate Number, the symbols referring to: Air Transport (AT), City Planning (CP), Construction (CO), Engineering Mechanics (EM), Highway (HW), Hydraulics (HY), Irrigation and Drainage (IR), Power (PO), Sanitary Engineering (SA), Soil Mechanics and Foundations (SM), Structural (ST), Surveying and Mapping (SU), and Waterways (WW) divisions. For titles and order coupons, refer to the appropriate issue of "Civil Engineering" or write for a cumulative price list.

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JANUARY: 379(SM)[c], 380(HY), 381(HY), 382(HY), 383(HY), 384(HY)[c], 385(SM), 386(SM), 387(EM), 388(SA), 389(SU)[c], 390(HY), 391(IR)[c], 392(SA), 393(SU), 394(AT), 395(SA)[c], 396(EM)[c], 397(ST)[c].

FEBRUARY: 398(IR)[d], 399(SA)[d], 400(CO)[d], 401(SM)[c], 402(AT)[d], 403(AT)[d], 404(IR)[d], 405(PO)[d], 406(AT)[d], 407(SU)[d], 408(SU)[d], 409(WW)[d], 410(AT)[d], 411(SA)[d], 412(PO)[d], 413(HY)[d].

MARCH: 414(WW)[d], 415(SU)[d], 416(SM)[d], 417(SM)[d], 418(AT)[d], 419(SA)[d], 420(SA)[d], 421(AT)[d], 422(SA)[d], 423(CP)[d], 424(AT)[d], 425(SM)[d], 426(IR)[d], 427(WW)[d].

APRIL: 428(HY)[c], 429(EM)[c], 430(ST), 431(HY), 432(HY), 433(HY), 434(ST).

MAY: 435(SM), 436(CP)[c], 437(HY)[c], 438(HY), 439(HY), 440(ST), 441(ST), 442(SA), 443(SA).

JUNE: 444(SM)[e], 445(SM)[e], 446(ST)[e], 447(ST)[e], 448(ST)[e], 449(ST)[e], 450(ST)[e], 451(ST)[e], 452(SA)[e], 453(SA)[e], 454(SA)[e], 455(SA)[e], 456(SM)[e].

JULY: 457(AT), 458(AT), 459(AT)[c], 460(IR), 461(IR), 462(IR), 463(IR)[c], 464(PO), 465(PO)[c].

AUGUST: 466(HY), 467(HY), 468(ST), 469(ST), 470(ST), 471(SA), 472(SA), 473(SA), 474(SA), 475(SM), 476(SM), 477(SM), 478(SM)[c], 479(HY)[c], 480(ST)[c], 481(SA)[c], 482(HY), 483(HY).

SEPTEMBER: 484(ST), 485(ST), 486(ST), 487(CP)[c], 488(ST)[c], 489(HY), 490(HY), 491(HY)[c], 492(SA), 493(SA), 494(SA), 495(SA), 496(SA), 497(SA), 498(SA), 499(HW), 500(HW), 501(HW)[c], 502(WW), 503(WW), 504(WW)[c], 505(CO), 506(CO)[c], 507(CP), 508(CP), 509(CP), 510(CP), 511(CP).

OCTOBER: 512(SM), 513(SM), 514(SM), 515(SM), 516(SM), 517(PO), 518(SM)[c], 519(IR), 520(IR), 521(IR), 522(IR)[c], 523(AT)[c], 524(SU), 525(SU)[c], 526(EM), 527(EM), 528(EM), 529(EM), 530(EM)[c], 531(EM), 532(EM)[c], 533(PO).

NOVEMBER: 534(HY), 535(HY), 536(HY), 537(HY), 538(HY)[c], 539(ST), 540(ST), 541(ST), 542(ST), 543(ST), 544(ST), 545(SA), 546(SA), 547(SA), 548(SM), 549(SM), 550(SM), 551(SM), 552(SA), 553(SM)[c], 554(SA), 555(SA), 556(SA), 557(SA).

DECEMBER: 558(ST), 559(ST), 560(ST), 561(ST), 562(ST), 563(ST)[c], 564(HY), 565(HY), 566(HY), 567(HY), 568(HY)[c], 569(SM), 570(SM), 571(SM), 572(SM)[c], 573(SM)[c], 574(SU), 575(SU), 576(SU), 577(SU), 578(HY), 579(ST), 580(SU), 581(SU), 582(Index).

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JANUARY: 583(ST), 584(ST), 585(ST), 586(ST), 587(ST), 588(ST), 589(ST)[c], 590(SA), 591(SA), 592(SA), 593(SA), 594(SA), 595(SA)[c], 596(HW), 597(HW), 598(HW)[c], 599(CP), 600(CP), 601(CP), 602(CP), 603(CP), 604(EM), 605(EM), 606(EM)[c], 607(EM).

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c. Discussion of several papers, grouped by Divisions.

d. Presented at the Atlanta (Ga.) Convention of the Society in February, 1954.

e. Presented at the Atlantic City (N.J.) Convention in June, 1954.